Subject

COUPLING OF HORIZONTAL AND VERTICAL OSCILLATIONS IN AN AGS

I. Introduction

As has been pointed out by Regenstreif and others,[1] the fringing field of quadrupole magnets causes a coupling between motion in the two transverse directions. This coupling must be considered in the design of a transport system, since it is capable of distorting the phase space in each of the two transverse directions and can result in a deterioration of useful beam quality.

This coupling also appears to be important in linear accelerators, where the particles enter and leave quadrupole magnets many times. The linac problem has been treated both numerically and analytically[2] and it is shown that while the amplitude of oscillation in either transverse direction may increase, the radial amplitude will be approximately constant. As a result it appears that no allowance for increased bore is necessary in a linac due to this coupling.

The present note is an application of these considerations to an AGS synchrotron, where an exchange of oscillation "energy" from radial to vertical has more serious consequences. In this case, however, it will turn out that inequality between the two transverse oscillation frequencies restricts this "energy exchange" to negligible values.

Comparison is also made with the amplitude growths resulting from angular misalignment of quadrupole axes. These appear to be far more serious and suggest as large a detuning of the horizontal and vertical betatron frequencies as practical.

II. Calculation of Amplitude Changes

It can readily be shown[2] that a particle entering and leaving a quadrupole magnet receives the following impulses per magnet.

$$\begin{aligned} \delta x' &\simeq -\frac{K_0^2 \ell}{4}\,(3xz^2 + x^3), \\ \delta z' &\simeq -\frac{K_0^2 \ell}{4}\,(3z^2 x + z^3), \end{aligned} \tag{1}$$

$$\delta x \simeq \delta z \simeq 0. \tag{2}$$

Here ℓ is the equivalent length of the magnet, and K_0 is proportional to the field gradient, i.e. the equation of motion within the magnet is

$$\begin{aligned} x'' + K_0 x &= 0, \\ z'' - K_0 z &= 0. \end{aligned} \tag{3}$$

In the "smooth" approximation one can write

$$\begin{aligned} x &= A \sin(k_x s + \alpha) \\ x' &= A k_x \cos(k_x s + \alpha) \\ z &= B \sin(k_z s + \beta) \end{aligned} \tag{4}$$

$$z' = Bk_z \cos(k_z s + \beta). \tag{4}$$

Here $k_x = \nu_x/R$, $k_z = \nu_z/R$ are the wave numbers of the horizontal and vertical betatron oscillations respectively. The changes in amplitude and phase per magnet can be obtained by averaging out rapidly varying terms. The result is

$$\delta A \simeq \frac{3}{32} \frac{K_0^2 \ell}{k_x} AB^2 \sin\Psi, \tag{5a}$$

$$\delta B \simeq -\frac{3}{32} \frac{K_0^2 \ell}{k_z} A^2 B \sin\Psi, \tag{5b}$$

$$\delta\alpha \simeq \frac{3}{32} \frac{K_0^2 \ell}{k_x} [B^2(2+\cos\Psi) + A^2], \tag{5c}$$

$$\delta\beta \simeq \frac{3}{32} \frac{K_0^2 \ell}{k_z} [A^2(2+\cos\Psi) + B^2], \tag{5d}$$

where

$$\Psi \simeq 2(k_x - k_z)s + 2\alpha - 2\beta, \tag{5e}$$

and where the "frequencies" k_x and k_z are taken to be almost the same.

Equations (5a) and (5b) lead immediately to the invariant

$$\frac{A^2}{k_z} + \frac{B^2}{k_x} = C^2 \text{ (= constant)} \tag{6}$$

One then obtains equations for δA^2 and $\delta\Psi$:

$$\delta A^2 \simeq \frac{3}{16} K_0^2 \ell \sin \Psi \; A^2 \left(C^2 - \frac{A^2}{k_z} \right) \tag{7}$$

$$\delta \Psi \simeq 2(k_x - k_z)L + \frac{3}{16} K_0^2 \ell \left[\left(2 + \cos \Psi\right)\left(C^2 - \frac{2A^2}{k_z}\right) + \frac{k_x^2 + k_z^2}{k_x k_z^2} A^2 - \frac{k_x}{k_z} C^2 \right], \tag{8}$$

where L is the distance between magnets.

It will be shown following Eq. (10) that the dominant term in Eq. (8) for an AGS is the first term on the right. In this case one can then write

$$\frac{\delta A^2}{A^2} \simeq -\frac{3}{16} K_0^2 \ell \frac{B^2}{k_x} \frac{\delta \cos \Psi}{2(k_x - k_z)L} . \tag{9}$$

The complicated invariant which can be constructed from Eqs. (7) and (8) restricts the excursion of A and B. This is simply approximated by setting $\delta \cos \Psi = \pm 2$ in Eq. (9), in which case we find for the maximum excursion in A

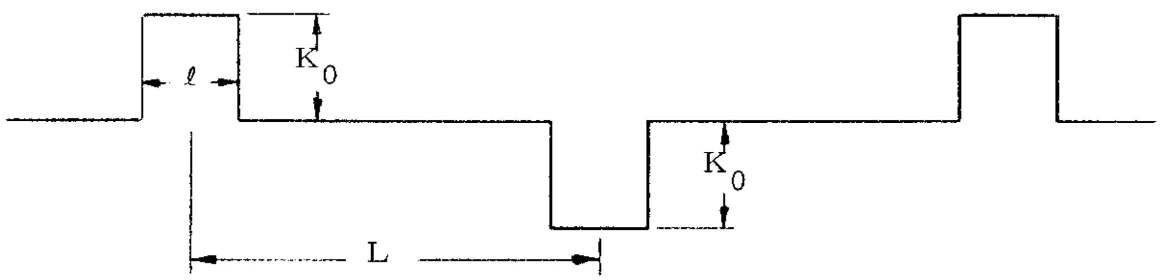
$$\left| \frac{\delta A}{A} \right| \simeq \frac{3}{32} \frac{K_0^2 \ell B^2}{k_x (k_x - k_z) L} . \tag{10a}$$

Similarly,

$$\left| \frac{\delta B}{B} \right| \simeq \frac{3}{32} \frac{K_0^2 \ell A^2}{k_z (k_x - k_z) L} . \tag{10b}$$

The quantities in Eq. (10a) or (10b) are precisely the relative magnitude of the neglected to the retained term in going from (8) to (9). If the amplitude changes in (10) are small, the approximation used in going from (8) to (9) is valid.

If we now take as our model for the transverse focusing



with $\ell << L$, we can write

$$\mu \simeq K_0 \ell L \,, \qquad k_x \simeq k_z \simeq \frac{K_0 \ell}{2} \,. \tag{11}$$

In the "smoothed" approximation

$$\left|\frac{\delta A}{A}\right| \simeq \frac{3}{8} \frac{B^2}{\ell L} \frac{\nu}{\Delta\nu} \,, \qquad \left|\frac{\delta B}{B}\right| \simeq \frac{3}{8} \frac{A^2}{\ell L} \frac{\nu}{\Delta\nu} \,, \tag{12}$$

where $\Delta\nu/\nu$ is fractional difference in horizontal and vertical "tunes."*

III. Numerical Values and Conclusions

The present parameters for the NAL main ring are

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*If one takes into account the strong focusing character of the oscillations, the result in Eq. (12) is reduced by the factor

$$\frac{\sin \mu}{\mu (1 + \sin \frac{\mu}{2})^2} \tag{12a}$$

where $\sin \frac{\mu}{2} = \frac{K_0 \ell L}{2}$.

$$\ell \simeq 2 \text{ meters}$$

$$L \simeq 30 \text{ meters}$$

$$A \leq 4 \text{ cm}$$

$$B \leq 2 \text{ cm}$$

$$\Delta\nu \sim .04$$

$$\nu \sim 20$$

With these parameters one finds from Eq. (12)

$$\left|\frac{\delta A}{A}\right| \leq 1.2 \times 10^{-3}, \qquad \left|\frac{\delta B}{B}\right| \leq 5 \times 10^{-3}. \tag{13}$$

These amplitude changes are clearly quite small* and can be neglected, as long as $\Delta\nu$ is not is not too small.

It is useful to compare the amplitude changes in Eq. (12) with those which occur because of random misalignment of the quadrupole axes. A simplified analysis with approximate results is given in the next section.

IV. Random Misalignment of Quadrupole Axes

Misalignment of the quadrupole axes can also lead to xy coupling. For axes rotated by a small angle θ one has

$$\begin{aligned} H_{x'} &= \alpha z' & \qquad H_x &\simeq \alpha(z + 2\theta x) \\ H_{z'} &= \alpha x' & \qquad H_z &\simeq \alpha(x - 2\theta z) \end{aligned} \tag{14}$$

and the equations of motion are

*The factor in (12a) reduces these changes to approximately 25% of the values in (13).

$$x'' + K(s)x = 2\,\theta(s)K(s)z$$
$$z'' - K(s)z = -2\,\theta(s)K(s)x \tag{15}$$

The term of greatest importance for $\nu_z \simeq \nu_x$ comes from that harmonic of the right side of (15) which corresponds roughly to the average of $2\theta(s)K(s)$. Retaining only this term, and using the "smoothed" approximation for the left side of (15), one obtains

$$x'' + (\nu_x{}^2/R^2)x \simeq \epsilon z\ ,$$
$$z'' + (\nu_z{}^2/R^2)z \simeq \epsilon x\ , \tag{16}$$

where

$$\epsilon = \frac{2}{2\pi R}\int_0^{2\pi R}\theta(s)K(s)\,ds = \frac{2K_0\ell}{2\pi R}\sum_{n=1}^{N}\theta_n(-1)^n = \frac{2K_0\ell}{L}\,\frac{1}{N}\sum_{n=1}^{N}\theta_n(-1)^n \tag{17}$$

Here N is the number of quadrupoles and the quadrupole geometry is that shown in the previous figure.

It is possible to solve Eq. (16) exactly. However, it is also useful to consider a phase amplitude method on (16) which leads to two adiabatic invariants. The first of these describes the fact that $A^2 + B^2$ is approximately constant, and the second restricts the excursions of A and B. This method also works if ϵ is a general harmonic of the rotation errors considered as a function of azimuth. In this case the exact solution is not easily obtained, but the adiabatic invariants appear as before. For sum-type resonances the second invariant leads to the usual stop band region in which the amplitudes grow without limit.

Our concern here is to estimate the amplitude growth corresponding to Eq. (16) in the vicinity of the $\nu_z \simeq \nu_x$ resonance. The perturbed part of the solution of Eq. (16) to first order in ϵ can be written as (note that $\delta x(0) = \delta x'(0) = 0$):

$$\delta x(s) \simeq \frac{\epsilon B \left[\cos(\nu_z s/R) - \cos(\nu_x s/R) \right]}{\left(\frac{\nu_x}{R}\right)^2 - \left(\frac{\nu_z}{R}\right)^2}, \tag{18}$$

and the resulting maximum excursion in x is

$$\left| \delta A \right| \simeq \frac{2\epsilon R^2}{\nu_x^2 - \nu_z^2} B \simeq \frac{\epsilon R^2}{\nu \Delta\nu} B , \tag{19a}$$

where $\Delta\nu = \nu_x - \nu_z$. Similarly,

$$\left| \delta B \right| \simeq \frac{\epsilon R^2}{\nu \Delta\nu} A . \tag{19b}$$

These formulas are valid as long as $\epsilon R^2/\nu\Delta\nu$ is small compared to 1. One can also expeess Eq. (19) in terms of the equivalent stop band width which, for the sum resonances, is*

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*This result for the amplitude growth is similar to that obtained by Courant and Snyder, Annals of Physics 3, (1958), in Eq. (4.55) for the one dimensional problem. The fact that our two dimensional result yields the same coefficient as their one dimensional result is related to the fact that coupling term is linear in the oscillation amplitude in both cases.

$$\delta \nu_{sB} = \frac{2\epsilon R^2}{\nu} , \qquad (20)$$

giving

$$\left| \frac{\delta A}{B} \right| \approx \left| \frac{\delta B}{A} \right| \approx \frac{\delta \nu_{sB}}{2\Delta\nu} \qquad (21)$$

For the lattice shown in the previous figure

$$\nu \approx \frac{N\mu}{4\pi} \approx \frac{K_0 \ell R}{2} , \qquad (22)$$

using Eq. (11). If one considers the angular errors to be randomly distributed, one can obtain an rms average for ϵ, leading to

$$\delta \nu_{sB} \approx \frac{4}{\pi} \theta_{rms} \sqrt{N} \qquad (23)$$

and*

$$\left| \frac{\delta A}{B} \right| \approx \left| \frac{\delta B}{A} \right| \approx \frac{2}{\pi} \frac{\theta_{rms} \sqrt{N}}{\Delta\nu} . \qquad (24)$$

Using

$$N = 200$$
$$\theta_{rms} = 10^{-3} \text{ rad} \qquad (25)$$
$$\Delta\nu = .04$$

one finds

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*The result in Eq. (23) is the same as that of Eq. (4.95) of Courant and Snyder (loc. cit.), applied to our quadrupole configuration.

$$\left|\frac{\delta A}{B}\right| \simeq \left|\frac{\delta B}{A}\right| \simeq .2 \tag{26}$$

If one desires to limit amplitude growths to less than 20%, θ_{rms} will have to be less than a milliradian, or $\Delta\nu$ will have to be increased, or both.

It is also possible to reduce the effect of this coupling by careful alignment of magnets axes in F-D pairs. If θ_1 is the rms value of the angle between the F and D axes, θ_{rms} in Eq. (24) is to be replaced by $\theta_1/\sqrt{2}$. If θ_2 is the rms value of the alignment of the pair, θ_{rms} in Eq. (24) is to be replaced by (27)

$$(\theta_1/2)\sqrt{\left(\frac{\beta_x{}^F}{\beta_y{}^F} - \frac{\beta_x{}^D}{\beta_y{}^D}\right)^2 + \left(\frac{\beta_x{}^F - \beta_y{}^F + \beta_x{}^D - \beta_y{}^D}{\beta_x\beta_y} \cdot L\right)^2}$$

where L is the separation of the doublet. It appears that the booster is being designed to have a small value of θ_1 in order to minimize the growth due to this coupling.

V. Conclusions

Coupling of the two transverse motions in the quadrupole fringing fields leads to the maximum amplitude growth given in Eqs. (12) and (12a). For the contemplated parameters of the NAL 200 BeV main ring, these growths are well below 1%.

These growths are evidently much less serious than those occurring because of angular misalignment of the quadrupoles, which are given in Eq. (24). In order to keep the maximum growths small, one must either detune ν_x and ν_z appropriately, or maintain a tight tolerance on the angular alignment of the quadrupoles.

VI. Acknowledgment

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FOOTNOTE AND REFERENCES

*Permanent address: Department of Physics and Astronomy, University of Massachusetts, Amherst, Massachusetts.

[1] E. Regenstreif, Proceedings of the Los Alamos Linac Conference, October 1966, p. 245; Proceedings of the Brookhaven Linac Conference, May 1968; P. W. Allison, Los Alamos Internal Report MP-4/PWA-1, July 27, 1967; P. W. Allison and R. R. Stevens, Proceedings of the Brookhaven Linac Conference, May 1968.

[2] R. Chasman, Proceedings of the Los Alamos Linac Conference, October 1966, p. 224; R. L. Gluckstern, Proceedings of the Los Alamos Linac Conference, October 1966, p. 250; Gluckstern, Stevens, and Allison, Los Alamos Internal Report MP-DO/2, August 1967.